

CALCULUS II TUTORIAL (8): SOLUTIONS

$$\textcircled{1} \quad \frac{\partial W}{\partial r} = \frac{\partial W}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial W}{\partial y} \frac{\partial y}{\partial r} = f_x \cos \theta + f_y \sin \theta \quad \textcircled{1}$$

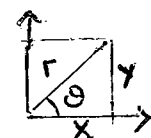
$$\frac{\partial W}{\partial \theta} = \frac{\partial W}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial W}{\partial y} \frac{\partial y}{\partial \theta} = f_x (-r \sin \theta) + f_y (r \cos \theta) \quad \textcircled{2}$$

$$\sin \theta \times \textcircled{1} + \frac{\cos \theta}{r} \times \textcircled{2} \Rightarrow f_y = W_r \sin \theta + \frac{1}{r} W_\theta \cos \theta$$

$$\cos \theta \times \textcircled{1} - \frac{\sin \theta}{r} \times \textcircled{2} \Rightarrow f_x = W_r \cos \theta - \frac{1}{r} W_\theta \sin \theta$$

$$\Rightarrow (f_x)^2 + (f_y)^2 = (W_r^2 \sin^2 \theta + \frac{1}{r^2} W_\theta^2 \cos^2 \theta + \frac{2}{r} W_r W_\theta \sin \theta \cos \theta) + (W_r^2 \cos^2 \theta + \frac{1}{r^2} W_\theta^2 \sin^2 \theta - \frac{2}{r} W_r W_\theta \cos \theta \sin \theta) = (W_r)^2 + \frac{1}{r^2} (W_\theta)^2$$

using:
 $x = r \cos \theta$
 $y = r \sin \theta$



$$\textcircled{2} \quad W = \frac{x}{z} + \frac{y}{z} = t \cos^2 t + t \sin^2 t = t \Rightarrow \frac{dW}{dt} = 1 \quad (\text{indpt of } t)$$

$$\text{Also } \frac{dW}{dt} = \frac{\partial W}{\partial x} \frac{dx}{dt} + \frac{\partial W}{\partial y} \frac{dy}{dt} + \frac{\partial W}{\partial z} \frac{dz}{dt} \quad \leftarrow \text{using } \cos^2 t + \sin^2 t = 1$$

$$= \frac{1}{z} \frac{d}{dt}(\cos^2 t) + \frac{1}{z} \frac{d}{dt}(\sin^2 t) - \left(\frac{x+y}{z^2}\right) \frac{d}{dt}\left(\frac{1}{t}\right) = t \frac{d(t)}{dt} - t^2 \frac{d(1/t)}{dt} = 1$$

$$\textcircled{3} \quad W = xy + yz + xz, \quad x = u + v, \quad y = u - v, \quad z = uv.$$

$$\text{Then } \frac{\partial W}{\partial u} = \frac{\partial W}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial W}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial W}{\partial z} \frac{\partial z}{\partial u} = (y+z)(1) + (x+z)(1) + (y+x)v$$

$$\frac{\partial W}{\partial v} = \frac{\partial W}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial W}{\partial y} \frac{\partial y}{\partial v} + \frac{\partial W}{\partial z} \frac{\partial z}{\partial v} = (u-v) + (u+v) + 2uv + (u-v+u+v)v = 2u + 4uv$$

$$= (y+z)(1) + (x+z)(-1) + (y+x)u = (u-v-u-v) + (u-v+u+v)u = -2v + 2u^2$$

$$\text{Also } W = (u+v)(u-v) + uv(u-v) + uv(u+v)$$

$$= u^2 - v^2 + 2u^2v \Rightarrow \frac{\partial W}{\partial u} = 2u + 4uv, \quad \frac{\partial W}{\partial v} = -2v + 2u^2$$

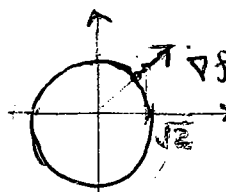
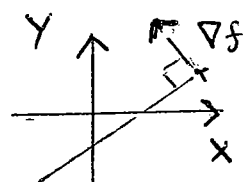
$\textcircled{4}$

at (2,1) is

$$(a) \quad f = y - x \Rightarrow \nabla f = (-1, 1), \quad \text{level curve } y - x = -1.$$

$$(b) \quad f = \ln(x^2 + y^2) \Rightarrow \nabla f = \frac{(2x, 2y)}{x^2 + y^2} = (1, 1) \quad \text{at } (1, 1).$$

$$\text{The level curve at } (1, 1) \text{ is } x^2 + y^2 = 2.$$



⑤

(a) $f = 2xy - 3y^2 \Rightarrow \nabla f = (2y, 2x - 6y) = (10, -20)$

$\underline{A} = (4, 3) \Rightarrow \hat{\underline{A}} = \left(\frac{4}{5}, \frac{3}{5}\right) \Rightarrow \nabla f \cdot \hat{\underline{A}} = \frac{40}{5} - \frac{60}{5} = -4$ at $(1, 4, 2)$
unit vector

(b) $f = xy + yz + zx \Rightarrow \nabla f = (y+z, x+z, y+x) = (1, 3, 0)$

$\underline{A} = (3, 6, -2) \Rightarrow |\underline{A}| = \sqrt{3^2 + 6^2 + 2^2} = 7 \Rightarrow \hat{\underline{A}} = \left(\frac{3}{7}, \frac{6}{7}, -\frac{2}{7}\right)$

$\Rightarrow \nabla f \cdot \hat{\underline{A}} = \frac{1 \cdot 3}{7} + \frac{3 \cdot 6}{7} - \frac{0 \cdot 2}{7} = 3$

⑥ $f = x^2 + xy + y^2 \Rightarrow \nabla f = (2x+y, x+2y) = (-1, 1)$ at $(-1, 1)$

$\uparrow$ most rapidly in dirn $\frac{\nabla f}{|\nabla f|} = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ with deriv $\sqrt{2}$

$\downarrow$ most rapidly in dirn $-\frac{\nabla f}{|\nabla f|} = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ with deriv $-\sqrt{2}$

⑦

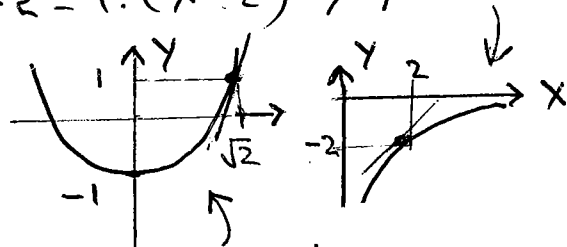
a) $xy = -4$ is level curve of $f = xy \Rightarrow \frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{y}{x}$

At $(2, -2): \frac{dy}{dx} = 1 \Rightarrow$ tangent is $y+2 = 1 \cdot (x-2) \Rightarrow y-x = -4$ *

b) $x^2 - y = 1$ is level curve of $f = x^2 - y$

$\Rightarrow \frac{dy}{dx} = -\frac{f_x}{f_y} = 2x = 2\sqrt{2}$ at $(\sqrt{2}, 1)$

Tangent is $y-1 = 2\sqrt{2}(x-\sqrt{2}) \Rightarrow y-2\sqrt{2}x = -3$ *



* Alternatively: $f_x(x_0, y_0)(x-x_0) + f_y(x_0, y_0)(y-y_0) = 0$ etc.

⑧

a) $x^2 + y^2 + z^2 = 3$ is level surface of $f = x^2 + y^2 + z^2$

$\Rightarrow \nabla f = (2x, 2y, 2z) = (2, 2, 2)$ at $(1, 1, 1)$

Tangent plane is $2(x-1) + 2(y-1) + 2(z-1) = 0 \Rightarrow x+y+z = 3$

Normal at $(1, 1, 1)$ is $x = 1+2t, y = 1+2t, z = 1+2t$

b) $2z - x^2 = 0$ is level surface of $f = 2z - x^2$

$\Rightarrow \nabla f = (-2x, 0, 2) = (-4, 0, 2)$ at $(2, 0, 2)$

Tangent plane is $-4(x-2) + 0(y-0) + 2(z-2) = 0 \Rightarrow 2x - z = 2$

Normal at $(2, 0, 2)$ is $x = 2-4t, y = 0, z = 2+2t$